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H^p Continuity Properties of Calderón–Zygmund-type Operators

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INTRODUCTION

In a recent paper (cf. [8]). G. David and J. L. Journé obtained a characterization of the Calderón–Zygmund operators, in the sense of R. Coifman and Y. Meyer (cf. [6]). These operators are an interesting generalization of the classical singular integral operators of Calderón–Zygmund. In particular, they include some classes of pseudo-differential operators, Cauchy integrals on Lipschitz curves, commutators of order n, etc.

The L^p theory of these operators is now well understood (cf. [8]). In this paper we consider the H^p theory (in the sense of Coifman and Weiss [7]) and obtain the following result:

THEOREM 1.1. *Let T be a δ-$C-Z$ operator* (*see Sect.* 1) *such that $T^*(1)=0$. Then, T extends to a continuous operator from H^p into itself for $1 \geqslant p > n/(n+\delta)$.*

The proof of this theorem consists in showing that the image under T of a (p, ∞)-atom is what we call a (p, q, α)-molecule (see Definition 1.1).

In the second part of the paper we introduce the class of strongly singular Calderón–Zygmund (C–Z) operators. Our study of these operators is motivated by the multiplier operators whose symbols are given

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by $e(i|\xi|^a)/|\xi|^\beta$ away from the origin. These multiplier operators have been studied by several authors (cf. [13, 17, 15, 9, 14, 11, 5, 3]).

In Theorem 2.1 we show that our strongly singular C–Z operators map L^∞ into BMO, extending a well-known result of Fefferman and Stein (cf. [11]). Moreover, we show that a strongly singular C–Z operator T satisfying $T^*(1)=0$ acts continuously on H^p spaces for a certain range of values of p, $1 \geqslant p > p_0$ (cf. Theorem 2.2). For convolution operators on $\mathbb{R}$, R. Coifman (cf. [5]) has obtained the continuity for the critical index p_0. We are not able to prove H^{p_0} continuity in the general case. However, we single out a class of convolution operators of strongly singular type, on $\mathbb{R}^n$, for which a general version of Coifman's result holds (cf. Theorem 2.3).

In the last part of the paper we consider applications of our results to obtain the H^p continuity of certain classes of pseudo-differential operators. In fact, following a suggestion of Stein, we show that our class of strongly singular C–Z operators includes the pseudo-differential operators with symbols in $S^{-b}_{a,\delta}$ where $0<\delta \leqslant a<1$, $(1-a)\,(n/2) \leqslant b < n/2$. The case $\delta < a$ was considered by Fefferman [10].

1. δ-Calderón Zygmund Operators

Let $T\colon \mathscr{S} \to \mathscr{S}'$ be a bounded linear operator. Following [6] and [8], we will say that T is associated with a δ-standard kernel if there exists a function $k(x, y)$ defined and continuous away from the diagonal in $\mathbb{R}^{2n}$, such that

$$|k(x, y)| \leqslant \frac{c}{|x-y|^n}, \qquad x \neq y$$

$$|k(x, y) - k(x, z)| + |k(y, x) - k(z, x)| \leqslant c\,\frac{|y-z|^\delta}{|x-z|^{n+\delta}},$$

$$\text{if } 2|y-z| \leqslant |x-z| \qquad \text{for some } 0<\delta \leqslant 1,$$

$$(T(f), g) = \int k(x, y)\, f(y)\, g(x)\, dy\, dx \quad \text{for } f, g \in \mathscr{S} \text{ with disjoint supports.}$$

If T is associated with a δ-standard kernel, we will say following [6] that T is a δ-Calderón–Zygmund operator if T can be extended to be a continuous operator from L^2 into itself. Every symbol $p(x, \xi)$ in the class $S^0_{1,1}$ generates a pseudo-differential operator associated with a 1-standard kernel, but this operator is not always continuous in L^2 (see [4]). On the other hand, given an amplitude $a(x, y, \xi)$ in the class $S^0_{1,1}$ with continuous derivatives in the variables x, y, ξ up to the orders $[n/2]+1$, $[n/2]+1$, $n+2$, respectively, the truncated integrals

$$L_\varepsilon(f)(x)=\int e^{-2\pi i(x-y)\xi}a(x,\,y,\,\xi)\,\eta(\varepsilon\xi)\,f(y)\,dy\,d\xi$$

converge in L^2 as $\varepsilon\to 0$, to a 1–C–Z operator (see [2] and [1]).

It is known (see [8]), that given an operator T associated with a δ-standard kernel and a function $f\in L^\infty\cap C^\infty$, $T(f)$ has a sense as a continuous linear functional on $\mathscr{D}_0=\{\varphi\in C_0^\infty/\int\varphi=0\}$. Moreover, as is proved in [6], if T is a δ–C–Z operator, it can be defined as a bounded operator from L^∞ to BMO. It is clear that if T is associated with a δ-standard kernel $k(x,y)$, T^* is associated with the δ-standard kernel $\overline{k(y,x)}$. Thus, T and T^* are simultaneously δ–C–Z operators.

THEOREM 1.1. *Let T be a δ–C–Z operator such that $T^*(1)=0$. Then, T extends to a continuous operator form H^p into itself, for $1\geqslant p>n/(n+\delta)$.*

As usual, when proving this kind of result, we will give an appropriate notion of molecule and we will show that these molecules belong to H^p and that T maps (p,∞)-atoms into molecules.

DEFINITION 1.1. Suppose $0<p\leqslant 1<q<\infty$, $\alpha>n((q/p)-1)$. A function $M(x)$ is a (p,q,α)-molecule if there exists a ball $B(z,\sigma)$ such that the following conditions are satisfied

(M_1) $\int|M(x)|^q\,dx\leqslant c\sigma^{n(1-q/p)}$,

(M_2) $\int|M(x)|^q|x-z|^\alpha\,dx\leqslant c\sigma^{\alpha+n(1-q/p)}$,

(M_3) $\int M(x)\,dx=0$.

Actually, to give a sense to condition (M_3), we will prove

LEMMA 1.1. *Let $M(x)$ be a function satisfying conditions* (M_1) *and* (M_2) *in Definition* 1.10. *Then, M is absolutely integrable.*

Proof. From condition (M_1), we get

$$\int_{B(z,\sigma)}|M(x)|\,dx\leqslant c|B|^{1-1/p}.$$

Moreover, if q' denotes the conjugate exponent of q, the function $|x-z|^{-\alpha/q}$ is q'-integrable on $\mathbb{R}^n\setminus B(z,\sigma)$. So, according to (M_2), we can write

$$\int_{\mathbb{R}^n\setminus B(z,\sigma)}|M(x)|\,dx\leqslant\|M|x-z|^{\alpha/q}\|_q\cdot\|\,|x-z|^{-\alpha/q}\|_{q'}\leqslant C|B|^{1-1/p}$$

This concludes the proof of the lemma.

This lemma shows that a molecule behaves like a $(p, 1)$-atom, with respect to $B(z, \sigma)$.

LEMMA 1.2. *Let $M(x)$ be a (p, q, α)-molecule. Then, in the L^q sense, we have*

$$M(x) = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where a_j is a (p, q)-atom supported on $B(z, 2^{j+1}\sigma)$ and $\sum_j |\lambda_j|^p < \infty$.

Proof. The proof is quite standard and will be only outlined here. Let $B_j = B(z, 2^j\sigma)$, $j = 0, 1, 2,...,$ and let M_j be the mean value of $M(x)$ on C_j, where $C_0 = B_0$, $C_j = B_j - B_{j-1}$, $j = 1, 2,....$ If χ_{C_j} is the characteristic function of C_j, define $\alpha_j(x) = (M(x) - M_j)\ \chi_{C_j}(x)$. It is clear that $\operatorname{supp}(\alpha_j) \subset B_j$, $\int \alpha_j\, dx = 0$. Furthermore,

$$\|\alpha_j\|_q^q \leqslant 2^q \int_{C_j} |M(x)|^q\, dx.$$

Using condition (M_2), we readily see that

$$\|\alpha_j\|_q \leqslant C2^{-j[\alpha/q + n(1/q - 1/p)]}|B_j|^{1/q - 1/p}.$$

Thus, the series $\sum_j \alpha_j$ converges in L^q and α_j can be written as $\lambda_j a_j$, where a_j is a (p, q)-atom supported on B_j and $\sum |\lambda_j|^p < \infty$.

Moreover,

$$M(x)\, \chi_{B_m} - \sum_{j=0}^{m} \alpha_j = \sum_{j=0}^{m} M_j \chi_{C_j},$$

where the left-hand side converges in the L^q-norm to $M(x) - \sum_{j=0}^{\infty} \alpha_j$. To take care of the right-hand side, we define a sequence $\{\delta_j\}$, $j = -1, 0, 1, 2,...,$ by

$$\delta_{-1} = \int M(x)\, dx = 0, \quad \delta_j = \int_{R^n \setminus B_j} M(x)\, dx, \quad j = 0, 1, 2,....$$

Then,

$$\sum_{j=0}^{m} M_j \chi_{C_j} = \sum_{j=0}^{m} (\delta_{j-1} - \delta_j)|C_j|^{-1}\, \chi_{C_j} = \sum_{j=0}^{m-1} \beta_j - \delta_m |C_m|^{-1}\, \chi_{C_m},$$

where $\beta_j(x) = \delta_j(|C_{j+1}|^{-1}\, \chi_{C_{j+1}} - |C_j|^{-1}\, \chi_{C_j})$. Clearly, $\operatorname{supp}(\beta_j) \subset B_{j+1}$ and $\int \beta_j dx = 0$. Furthermore, we can estimate $\|\beta_j\|_q$ in the same way as $\|\alpha_j\|_q$, to obtain $\|\beta_j\|_q \leqslant C2^{-j[\alpha/q + n(1/q - 1/p)]}|B_{j+1}|^{1/q - 1/p}$.

Finally, $\delta_m|C_m|^{-1}\chi_{C_m} \to 0$ as $m \to \infty$ in the sense of $\mathscr{S}'$. In fact, if $\varphi \in \mathscr{S}$, we have $|\delta_m|C_m|^{-1}\int_{C_m}\varphi(x)\,dx| \leqslant C\|\varphi\|_\infty \cdot |\delta_m| \xrightarrow[m\to\infty]{} 0$, since $M \in L^1$. This completes the proof of the lemma.

THEOREM 1.2. *Let T be a δ–C–Z operator such that $T^*(1)=0$. Then, if a is a (p,∞)-atom supported on $B(z,\sigma)$, $n/(n+\delta)<p\leqslant 1$, $T(a)$ is a (p,q,α)-molecule related to $B(z,\sigma)$, if $1<q<\infty$, $n(q/p-1)<\alpha<q(n+\delta)-n$.*

Proof. We have to check that the function $T(a)$ satisfies the conditions of Definition 1.1. Using the fact that a δ–C–Z operator extends to a continuous operator from L^q into itself for $1<q<\infty$, we have

$$\int |T(a)|^q\,dx \leqslant C\|a\|_q^q \leqslant C\cdot\sigma^{n(1-q/p)}.$$

On the other hand,

$$\int_{B(z,2\sigma)} |T(a)|^q|x-z|^\alpha\,dx \leqslant C\sigma^\alpha \int |T(a)|^q\,dx \leqslant C\sigma^{\alpha+n(1-q/p)},$$

$$\int_{\mathbb{R}^n\setminus B(z,2\sigma)} |T(a)|^q|x-z|^\alpha\,dx = \sum_{j=0}^\infty \int_{B(z,2^{j+2}\sigma)\setminus B(z,2^{j+1}\sigma)} |T(a)|^q|x-z|^\alpha\,dx.$$

Moreover, since a is an atom supported on $B(z,\sigma)$, we have, for $|x-z|>2^{j+1}\sigma$,

$$T(a)(x)=\int [k(x,y)-k(x,z)]\,a(y)\,dy.$$

According to the definition of a δ-standard kernel, we have the estimate

$$|T(a)(x)| \leqslant C\int_{|y-z|\leqslant\sigma} \frac{|y-z|^\delta}{|x-z|^{n+\delta}}|a(y)|\,dy \leqslant C\sigma^{\delta+n(1-1/p)}|x-z|^{-n-\delta}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(z,2\sigma)} |T(a)|^q|x-z|^\alpha\,dx &\leqslant C\sum_{j=0}^\infty \sigma^{q[\delta+m(1-1/p)}\int_{2^{j+1}\sigma}^\infty \rho^{n+\alpha-q(n+\delta)-1}\,d\rho\\ &\leqslant C\sigma^{\alpha+n(1-q/p)}.\end{aligned}$$

Finally, since $T(a)\in L^1$ and $|B|^{-1+1/p}a$ is a $(1,\infty)$-atom, we get

$$0=(|B|^{-1+1/p}a,\,T^*(1))=(T(|B|^{-1+1/p}a),1)=|B|^{-1+1/p}\int T(a)\,dx.$$

This completes the proof of the theorem.

It is also possible to consider atomic spaces where higher moments vanish. But in that case, more regularity on the kernel $k(x, y)$ is required. In the case of a convolution operator, the condition $T^*(1) = 0$ in Proposition 1.1 can be dropped.

2. Strongly Singular Calderón–Zygmund Operators

Several authors (see [13, 17, 15, 9, 14]), have studied a class of multiplier operators whose symbol is given by $e^{i|\xi|^a}/|\xi|^\beta$ away from the origin, $0 < a < 1$, $\beta > 0$. Fefferman and Stein have enlarged this onto a class of convolution operators (see [11, pp. 142, 158]). Coifman has also considered in [5, p. 273], a related class of operators, for $n = 1$.

Motivated by these results, and the work of Macías, Segovia, and Bordin (cf. [3]) we introduce the following class of operators.

Definition 2.1. Let $T\colon \mathcal{S} \to \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón–Zygmund operator if the following conditions are fulfilled.

(S_1) T extends to a continuous operator from L^2 into itself.

(S_2) T is associated with a certain standard kernel. More precisely, there exists a function $k(x, y)$ continuous away the diagonal on $\mathbb{R}^{2n}$ such that

$$|k(x, y) - k(x, z)| + |k(y, x) - k(z, x)| \leqslant C \frac{|y - z|^\delta}{|x - z|^{n + \delta/\alpha}},$$

if

$$2|y - z|^\alpha \leqslant |x - z| \quad \text{for some } 0 < \delta \leqslant 1,\ 0 < \alpha < 1,$$

$(T, g) = \int k(x, y) f(y) g(x)\, dy\, dx$, for $f, g \in \mathcal{S}$ with disjoint supports.

(S_3) For some $(1 - \alpha)\, n/2 \leqslant \beta < n/2$, both operators T and T^*, extend to continuous operators from L^q into L^2, where $1/q = \frac{1}{2} + (\beta/n)$.

If T has a standard kernel in the sense of condition (S_2), given $f \in L^\infty \cap C^\infty$, $T(f)$ can be considered as a continuous linear functional on $\mathcal{D}_0$. The proof follows the one in [8] with minor changes and we shall omit it.

Theorem 2.1. *If T is a strongly singular C–Z operator, then T can be defined as a continuous operator from L^∞ into* BMO.

Proof. Fix a ball $B(z, \sigma)$ and consider first the case $\sigma \leqslant 1$. Given $f \in L^{\infty}$, write $f = f\chi_{B(z,2\sigma^{\alpha})} + f(1-\chi_{B(z,2\sigma^{\alpha})}) = f_1 + f_2$.

According to (S_3), T extends to a continuous operator from L^2 into $L^{q'}$, where $1/q' = 1-(1/q) = \frac{1}{2} - (\beta/n)$. Thus, $T(f_1)$ has a meaning and belongs to $L^{q'} \subset L^1(B(z, \sigma))$. Moreover,

$$\int_{B(z,\sigma)} |T(f_1)|\, dx \leqslant |B(z, \sigma)|^{1/q} \|T(f_1)\|_{q'} \leqslant C\|f\|_{\infty} |B(z, \sigma)|^{\alpha/2 + 1/q}.$$

Thus, since $(\alpha/2) + (1/q) - 1 \geqslant 0$,

$$\frac{1}{|B(z, \sigma)|} \int_{B(z,\sigma)} |T(f_1)|\, dx \leqslant C \cdot \|f\|_{\infty} |B(z, \sigma)|^{\alpha/2 + 1/q - 1} \leqslant C\|f\|_{\infty}.$$

When $|y-z| \geqslant 2\sigma^{\alpha}$, $|x-z| \leqslant \sigma$, we have $2|x-z|^{\alpha} \leqslant |y-z|$ and using (S_2), we get

$$|k(x, y) - k(z, y)| \leqslant \frac{c|x-z|^{\delta}}{|y-z|^{n+\delta/\alpha}}.$$

Therefore, the function $[k(x, y) - k(z, y)]\ f_2(y)$ is integrable on y, if $|x-z| \leqslant \sigma$. Moreover,

$$\int_{|y-z| \geqslant 2\sigma^{\alpha}} |k(x, y) - k(z, y)|\, |f(y)|\, dy \leqslant C\|f\|_{\infty}.$$

Let us now suppose that $\sigma > 1$. In this case, split $f \in L^{\infty}$ by

$$f_1 = f\chi_{B(z,2\sigma)}, \qquad f_2 = f(1-\chi_{B(z,2\sigma)}).$$

According to condition (S_1), we have

$$\frac{1}{|B(z,\sigma)|} \int_{B(z,\sigma)} |T(f_1)|\, dx \leqslant C|B(z,\sigma)|^{-1/2} \|f_1\|_2 \leqslant C\|f\|_{\infty}.$$

Now, if $|x-z| \leqslant \sigma$, $|y-z| \geqslant 2\sigma$, we get $2|x-z|^{\alpha} \leqslant 2\sigma^{\alpha} \leqslant 2\sigma \leqslant |y-z|$. Thus, we obtain,

$$\frac{1}{|B(z, \sigma)|} \int_{B(z,\sigma)} \left[\int_{|y-z| \geqslant 2\sigma} |k(x, y) - k(z, y)|\, |f(y)|\, dy \right] dx \leqslant C\|f\|_{\infty}.$$

as required.

This result extends Theorem 1 in [11]. We remark that only the continuity of T from L^2 to $L^{q'}$ was used.

To emphasize the fact that the operators we are dealing with are generalizations of convolution operators, it is sometimes convenient to denote the kernels by $k(x, x-y)$. Furthermore, condition (S_2) can be replaced by an integral one:

(S_2') $\int_{|x| \geqslant 2\sigma^\alpha} |k(x+z, x-y) - k(x+z, x)|\, dx + \int_{|x| \geqslant 2\sigma^\alpha} |k(x-y, x+z) - k(x, x+z)|\, dx \leqslant C$ for $|y| \leqslant \sigma$, $\sigma > 0$, $z \in \mathbb{R}^n$.

THEOREM 2.2. *Let T a strongly singular C–Z operator such that $T^*(1) = 0$. Let $1/p_0 = \frac{1}{2} + (\beta(\delta/\alpha + n/2)/n(\delta/\alpha - \delta + \beta))$. Then, if $1 \geqslant p > p_0$, T extends to a continuous operator from H^p into itself.*

Remark. This theorem includes a result stated in [11, p. 191]. To see this set $1/\alpha = \lambda - a' + 1$, $\delta = 1$, $\beta = b = \lambda(a-1) + (an/2)$. The H^{p_0}-continuity is proved in [5], for $n = 1$.

The critical case remains open. However, it is easy to adapt the 1-dimensional proof, to get an n-dimensional version of Coifman's result. In fact we have

THEOREM 2.3. *Let $k(x)$ be a function, continuous away from the origin, with $\operatorname{supp}(k) \subset \{|x| \leqslant 1\}$. Moreover, suppose that for some $0 < \beta, \delta, \eta, \theta$ such that $0 \leqslant \theta < \beta + \eta/\delta + \eta + (n/2)$, $(n/2) \cdot (\eta/(\eta + \delta)) \leqslant \beta < n/2$, $0 < \delta \leqslant 1$, we have*

$$|k(x-y) - k(x)| \leqslant C \frac{|y|^\delta}{|x|^{n+\delta+\eta}} \qquad \text{if} \quad 2|y|^{1-\theta} \leqslant |x|,$$

$$|\hat{k}(\xi)| \leqslant C(1 + |\xi|)^{-\beta}.$$

Then, $k*$ can be defined as a continuous operator from H^p into itself, for $1 > p \geqslant p_0$, where $1 < 1/p_0 = \frac{1}{2} + (\beta/n) \cdot ((\delta + \eta + n/2)/(\beta + \eta))$.

Proof. It suffices to show that $k*$ maps continuously H^{p_0} into L^{p_0} (cf. [11]). Let a be a (p_0, ∞)-atom supported on a ball $B(0, r)$. First, suppose that $r < 1$.

Let $\gamma < 0$ to be determined precisely later. Split $\int |k * a|^{p_0}\, dx$ as follows:

$$(\mathrm{I}) = \int_{|x| \leqslant 2r^\gamma}, \qquad (\mathrm{II}) = \int_{|x| \geqslant 2r^\gamma}.$$

To estimate (I), proceed as follows:

$$\int_{|x| \leqslant 2r^\gamma} |k * a|^{p_0}\, dx \leqslant C r^{n\gamma/(2/p_0)'} \cdot \|k * a\|_2^{p_0}.$$

The hypothesis over k imply that the convolution operator $k*$ is, in some sense, a slightly more general strongly singular C–Z operator. Nevertheless, using condition (S_3) we get

$$\|T(a)\|_2^{p_0} \leqslant C \cdot \|a\|_q^{p_0} \leqslant C \cdot |B|^{p_0/q-1},$$

where $1/q = \frac{1}{2} + (\beta/n)$.

Finally, we get

$$(\mathrm{I}) \leqslant C \cdot r^{n\gamma(2-p_0)/2 + np_0/2 + p_0\beta - n}.$$

On the other hand, if $|x| \geqslant 2r^\gamma$, since $\operatorname{supp}(a) \subset \{|y| \leqslant r\}$ and $r \leqslant 1$, we obtain

$$|k(x-y) - k(x)| \leqslant C \frac{|y|^\delta}{|x|^{n+\delta+\eta}}$$

whenever $0 < \gamma \leqslant 1 - \theta$. Therefore, using that $p_0 > n/(n+\delta+\eta)$ we obtain

$$\begin{aligned}(\mathrm{II}) &\leqslant \int_{|x| \geqslant 2r^\gamma} \left[\int_{|y| \leqslant r} |k(x-y) - k(x)|\, |a(y)|\, dy\right]^{p_0} dx \\ &\leqslant Cr^{p_0(n+\delta) - n + \gamma[n - p_0(n+\delta+\eta)]}.\end{aligned}$$

Our aim now is to choose γ in such a way that both exponents in the estimates of (I) and (II), vanish. It is easy to verify that we should define γ by

$$\gamma = \frac{p_0(n+\delta) - n}{p_0(n+\delta+\eta) - n}.$$

Moreover, taking into account the hypothesis on θ, we will also get the condition $0 < \gamma \leqslant 1 - \theta$. This gives the desired result when $r < 1$.

Next, suppose that $r \geqslant 1$. Split the integral $\int |k * a|^{p_0}\, dx$ in two terms:

$$(\mathrm{I}) = \int_{|x| \leqslant 2r}, \qquad (\mathrm{II}) = \int_{|x| > 2r}.$$

Since $(k * a)(x) = 0$ when $|x| > 2r$, the integral (II) vanishes.

On the other hand,

$$\int_{|x| \leqslant 2r} |k * a|^{p_0}\, dx \leqslant Cr^{n/(2/p_0)'} \|k * a\|_2^{p_0} \leqslant C \cdot r^{n(1-p_0/2)} \|a\|_2^{p_0} \leqslant C$$

since $k*$ is a bounded operator from L^2 into itself. This concludes the proof of the theorem.

As in Section 1, to prove Theorem 2.2, we need an adequate notion of molecule. Since the number of parameters involved in the present case is larger, it seems better to start with the following lemma (see [3]).

LEMMA 2.1. *Let T be a strongly singular C–Z operator satisfying the hypothesis in Theorem 2.2. Let a be a $(p, 2)$-atom supported on $B(z, \sigma)$ and finally, let $M(x) = T(a)(x)$. Then, $M(x)$ satisfies the following conditions: If $\sigma > 1$,*

(M_1) $\int |M(x)|^2\, dx \leqslant C\sigma^{n(1-2/p)}$.

(M_2) $\int |M(x)|^2 |x-z|^\lambda\, dx \leqslant C\sigma^{\lambda+n(1-2/p)}$, *for some* $(2n/p) - n < \lambda < n + (2\delta/\alpha)$. *If* $\sigma \leqslant 1$,

(M_1') $\int |M(x)|^2\, dx \leqslant C\sigma^{2n(1/q-1/p)}$.

(M_2') $\int |M(x)|^2 |x-z|^\lambda\, dx \leqslant C\sigma^{\rho\lambda+2n(1/q-1/p)}$, *for some* $(2n/p) - n < \lambda < (2\beta/(1-\rho)) \leqslant n + (2\delta/\alpha)$,

$$\rho = \frac{n/2 + \delta - \beta}{n/2 + \delta/\alpha} \leqslant \alpha.$$

$M(x)$ is an integrable function such that $\int M(x)\, dx = 0$.

Proof. To prove $M \in L^1$, we proceed as in Lemma 1.1. First, suppose that $\sigma > 1$. Then,

$$\int_{B(z,\sigma)} |M(x)|\, dx \leqslant C|B|^{1-1/p}.$$

On the other hand, since $\lambda > n$

$$\int_{\mathbb{R}^n \setminus B(z,\sigma)} |M(x)|\, dx \leqslant \|M(x) \cdot |x-z|^{\lambda/2}\|_2 \cdot \sigma^{(n-\lambda)/2} \leqslant C \cdot |B|^{1-1/p}.$$

When $\sigma < 1$, we use conditions (M_1') and (M_2'),

$$\int_{B(z,\sigma)} |M(x)|\, dx \leqslant C|B|^{1/q-1/p+1/2} = C|B|^{\beta/n+1-1/p},$$

$$\int_{\mathbb{R}^n \setminus B(z,\sigma)} |M(x)|\, dx \leqslant C\sigma^{\rho\lambda/2+n(1/q-1/p)+(n-\lambda)/2} = C|B|^{(\rho-1)\lambda/2n+(1/q-1/p+1/2)}$$
$$= C|B|^{\beta/n+1-1/p-(1-\rho)\lambda/2n}.$$

Since the operator T can be defined as a continuous operator from L^∞ into BMO, we can prove that $\int M(x)\,dx = 0$, as in Proposition 1. Now we prove the conditions stated at the beginning of the lemma. Suppose $\sigma > 1$.

$$\int |M(x)|^2\,dx \leqslant C \cdot \|a\|_2^2 \leqslant C \cdot \sigma^{n(1-2/p)},$$

$$\begin{aligned}\int |M(x)|^2 |x-z|^\lambda\,dx &\leqslant \int_{B(z,2\sigma^\alpha)} |M(x)|^2 |x-z|^\lambda\,dx \\ &\quad + \int_{\mathbb{R}^n \setminus B(z,2\sigma^\alpha)} |M(x)|^2 |x-z|^\lambda\,dx \\ &= (\mathrm{I}) + (\mathrm{II}),\end{aligned}$$

$$(\mathrm{I}) \leqslant C\sigma^{\lambda + n(1-2/p)} \qquad \text{since} \quad \alpha < 1.$$

To estimate (II), we proceed as in Proposition 1,

$$(\mathrm{II}) = \sum_{j=0}^{\infty} \int_{B(z,2^{j+2}\sigma^\alpha) \setminus B(z,2^{j+1}\sigma^\alpha)} |M(x)|^2 |x-z|^\lambda\,dx.$$

If $2|y-z|^\alpha \leqslant 2\sigma^\alpha \leqslant 2^{j+1}\sigma^\alpha \leqslant |x-z|$, we get the estimate

$$\begin{aligned}|M(x)|^2 &\leqslant \left[\int_{|y-z|\leqslant\sigma} |k(x,y)-k(x,z)|\,|a(y)|\,dy\right]^2 \\ &\leqslant C \frac{\sigma^{2\delta}}{|x-z|^{2(n+\delta/\alpha)}} \left(\int_{|y-z|\leqslant\sigma} |a(y)|\,dy\right)^2 \\ &\leqslant C \frac{\sigma^{2\delta+2n(1-1/p)}}{|x-z|^{2(n+\delta/\alpha)}}.\end{aligned}$$

Thus, since $\lambda < n + (2\delta/\alpha)$,

$$(\mathrm{II}) \leqslant C \sum_{j=0}^{\infty} \sigma^{2\delta+2n(1-1/p)} (2^j\sigma^\alpha)^{\lambda - 2\delta/\alpha - n} \leqslant C \cdot \sigma^{\alpha\lambda + 2n(1-1/p) - n\alpha}.$$

This can be majorized by $C\sigma^{\lambda + n(1-2/p)}$, since $\alpha < 1$, $\lambda > n$.

Now, suppose that $\sigma \leqslant 1$.

$$\int |M(x)|^2\,dx \leqslant C \cdot \|a\|_q^2 \leqslant C\sigma^{2n(1/q-1/p)},$$

$$\int |M(x)|^2 |x-z|^\lambda\,dx = \int_{B(z,2\sigma^\rho)} + \int_{\mathbb{R}^n \setminus B(z,2\sigma^\rho)} = (\mathrm{I}) + (\mathrm{II}).$$

We have

$$(\mathrm{I}) \leqslant C\sigma^{\rho\lambda + 2n(1/q - 1/p)},$$

$$(\mathrm{II}) = \sum_{j=0}^{\infty} \int_{B(z, 2^{j+2}\sigma^{\rho}) \setminus B(z, 2^{j+1}\sigma^{\rho})} |M(x)|^2 |x - z|^{\lambda}\, dx.$$

Since, $2|y - z|^{\alpha} \leqslant 2\sigma^{\alpha} \leqslant 2\sigma^{\rho} \leqslant |x - z|$, we can use the same estimate as above. Thus,

$$(\mathrm{II}) \leqslant C \cdot \sum_{j=0}^{\infty} \sigma^{2\delta + 2n(1 - 1/p)} (2^j \sigma^{\rho})^{\lambda - n - 2\delta/\alpha} \leqslant C \cdot \sigma^{2\delta + 2n(1 - 1/p) + \rho\lambda - \rho(n + 2\delta/\alpha)}$$

$$= C \cdot \sigma^{\rho\lambda + 2n(1/q - 1/p)} \qquad \text{since} \quad \rho = \frac{n(1 - 1/q) + \delta}{n/2 + \delta/\alpha}.$$

This completes the proof of the lemma.

DEFINITION 2.2. Given a function $M(x)$, we say that M is a (p, ρ, λ)-molecule related to $B(z, \sigma)$, when M satisfies conditions (M_1), (M_2) if $\sigma > 1$ or conditions (M_1'), (M_2') if $\sigma \leqslant 1$ (see Lemma 2.1) and moreover, $\int M\, dx = 0$.

LEMMA 2.2. *Let $M(x)$ be a (p, ρ, λ)-molecule. Then, in the L^2 sense,*

$$M(x) = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where a_j is a $(p, 2)$-atom supported by $B(z, 2^{j+1}\sigma)$ and $\sum_j |\lambda_j|^p < \infty$.

Proof. First, suppose that $\sigma > 1$. Since conditions (M_1) and (M_2) in Lemma 2.1 are exactly the conditions in Definition 1.1 for $\alpha = \lambda$, $q = 2$, there is nothing to prove.

Suppose, then, that $\sigma \leqslant 1$. With the notation of Lemma 1.2, we have

$$M(x)\, \chi_{B_m} = \sum_{j=0}^{m} \alpha_j + \sum_{j=0}^{m-1} \beta_j - \delta_m |C_m|^{-1} \chi_{C_m},$$

$$\|\alpha_j\|_2^2 \leqslant 4 \cdot \int_{C_j} |M(x)|^2\, dx.$$

According to condition (M_2'), we have

$$\|\alpha_j\|_2 \leqslant C \cdot (2^j \sigma)^{-\lambda/2} \sigma^{\rho\lambda/2 + n(1/q - 1/p)}.$$

Since the exponent of 2^j is negative, the series $\sum_j \alpha_j$ converges in L^2. Moreover, we can also write

$$\|\alpha_j\|_2 \leqslant C|B_j|^{1/2-1/p}(2^j\sigma)^{n(1/p-1/2)-\lambda/2}\sigma^{\rho\lambda/2+n(1/q-1/p)}.$$

The exponent of σ is $n(1/p-1/2) - \lambda/2 + \rho\lambda/2 + n(1/q-1/p) = -(\lambda/2)(1-\rho)+\beta$. Since $\sigma \leqslant 1$, we need $\lambda \leqslant 2\beta/(1-\rho)$. On the other hand, $\sum_{j=0}^{\infty} (2^j)^{n(1-p/2)-\lambda p/2} < \infty$, since $\lambda > (2n/p)-n$. $\|\beta_j\|_2$ can be estimated in the same way.

Finally, as in Lemma 1.2, $M(x)\chi_{B_m} \to M(x)$ as $m\to\infty$ in L^2 and $\delta_m |C_m|^{-1} \chi_{C_m} \to 0$ as $m \to \infty$ in $\mathscr{S}'$.

3. Applications

Following a suggestion of Stein, we will now show that the class of operators given by Definition 2.1 includes pseudo-differential operators with symbols in the class $S^{-b}_{a,\delta}$, where $0<\delta\leqslant a<1$, $(1-a)\,n/2 \leqslant b < n/2$. The case $\delta < a$ was considered in [10]. More generally, we consider amplitudes $p(x, y, \xi)$ in the class $S^{-b}_{a,\delta}$. If $\eta(\varepsilon\xi)$, $0<\varepsilon\leqslant 1$, denotes an usual truncation function, we can prove that

$$Lf(x) = \lim_{\varepsilon\to 0} \int e^{-2\pi i(x-y)\xi} p(x, y, \xi)\, \eta(\varepsilon\xi)\, f(y)\, dy\, d\xi, \quad f\in\mathscr{S},$$

exists in L^2 and that L extends to a continuous operator from L^2 into itself. (See [2]).

We are going to show that L is an strongly singular C–Z operator in the sense of Definition 2.1. More precisely, we will verify condition (S_2') instead of condition (S_2). We assume that p has compact support in ξ and vanishes for $|\xi| \leqslant 1$, thus our aim is to get an estimate that does not depend on the support of p.

Let ψ be a C_0^∞ function supported on $\{\frac{1}{2} \leqslant |\xi| \leqslant 2\}$, such that $\sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1$ for $|\xi| \geqslant 1$. Define

$$p_j(x, y, \xi) = p(x, y, \xi)\, \psi(2^{-j}\xi),$$

$$k_j(x, u) = \int e^{-2\pi i u\xi} p_j(x, x-u, \xi)\, d\xi,$$

$$K_m(x, u) = \sum_{j=0}^{m} k_j(x, u).$$

We will show that there exists a constant $C > 0$ such that

$$\int_{|x| \geq 2t^a} |K_m(x+z, x-y) - K_m(x+z, x)|\, dx \leq C \quad \text{for } z \in \mathbb{R}^n,$$
$$|y| \leq t,\ t > 0,\ m = 0, 1, 2, \ldots.$$

The proof is an adaptation of Lemma 3.3 in [16]. Let $N = [n/2] + 1$, $|\alpha| \leq N$.

$$(2^j x)^\alpha k_j(x+z, x) = (2^j x)^\alpha \int e^{-2\pi i x\xi} p(x+z, 0, \xi)\, \psi(2^{-j}\xi)\, d\xi$$
$$= 2^{j|\alpha|} \sum_{\beta \leq \alpha} C_\beta \int e^{-2\pi i x\xi} D_\xi^\beta p(x+z, 0, \xi) \cdot D^{\alpha-\beta} \psi(2^{-j}\xi)\, d\xi.$$

Since $\{|\xi|^{a|\beta| + b} D_\xi^\beta p(x+z, 0, \xi)\}$ is a bounded subset of $S^0_{a,\delta}$, with bounds not depending on z, we have the estimate

$$\|(2^j x)^\alpha k_j(x+z, x)\|_2 \leq C \cdot 2^{j|\alpha|} \sum_{\beta < \alpha} \|\, |\xi|^{-a|\beta| - b} D^{\alpha-\beta} \psi(2^{-j}\xi)\|_2$$
$$\leq C \cdot 2^{j[(1-a)N + an/2]}.$$

Thus,

$$\int_{|x| \geq 2t^a} |k_j(x+z, x)|\, dx \leq C 2^{j[(1-a)N + an/2]} \left(\int_{|x| \geq 2t^a} (2^j |x|)^{-2N}\, dx \right)^{1/2}$$
$$\leq C(2^j t)^{a(n/2 - N)}.$$

Furthermore, $k_j(x+z, x-y)$ can be handled in the same way, by considering $\int e^{-2\pi i x\xi} p_j(x+z+y, y, \xi) \cdot d\xi$. Therefore,

$$\int_{|x| \geq 2t^a} k_j(x+z, x-y) - k_j(x+z, x)|\, dx \leq C \cdot (2^j t)^{a(n/2 - N)}.$$

Now, we are going to improve this estimate when $2^j t \leq 1$. In this case we write

$$k_j(x+z, x-y) - k_j(x+z, x) = \int e^{-2\pi i x\xi} (e^{2\pi i y\xi} - 1)\, p_j(x+z, y, \xi)\, d\xi$$
$$+ \int e^{-2\pi i x\xi} [p_j(x+z, y, \xi) - p_j(x+z, 0, \xi)]\, d\xi.$$

Since $|e^{i\theta}-1| \leqslant |\theta| \leqslant |\theta|^s$ if $|\theta| \leqslant 1$, $0<s<1$, $|e^{i\theta}-1|/|\theta|^s \xrightarrow[|\theta|\to\infty]{} 0$, we observe that $|e^{2\pi i y\xi}-1| \leqslant C|y|^a|\xi|^a \leqslant C(2^jt)^a$, on the support of $\psi(2^{-j}\xi)$. $|D_\xi^\gamma e^{2\pi i y\xi}| \leqslant C|y|^{|\gamma|} \leqslant C2^{-ja|\gamma|}(2^jt)^{a|\gamma|} \leqslant C\cdot 2^{-ja|\gamma|}(2^jt)^a$, if $|\gamma|>0$. It follows that $\{|\xi|^{a|\beta|+b}D_\xi^\beta[(e^{2\pi i y\xi}-1)\ p(x+z, y, \xi)]\ \theta(2^{-j}\xi)\}$ is a bounded subset of $S^0_{a,\delta}$, with bounds that can be majorized by $(2^jt)^a$, where θ is a C_0^∞ function such that $\theta\psi=\psi$, $|\xi|\sim C$ on $\operatorname{supp}(\theta)$. On the other hand, using

$$p_j(x+z, y, \xi)-p_j(x+z, 0, \xi)=\sum_{l=1}^{n}\int_0^1 \frac{\partial p_j}{\partial y_l}(x+z, ty, \xi)\cdot dt\, y_l,$$

we see that

$$\begin{aligned}|D_\xi^\beta[p_j(x+z, y, \xi)-p_j(x+z, 0, \xi)]| &\leqslant C\cdot|\xi|^{-b+a-a|\beta|}|y|\\ &\leqslant C\cdot|\xi|^{-b-a|\beta|}\cdot(2^jt)^a\end{aligned}$$

on $\operatorname{supp}(\psi(2^{-j}\xi))$. Thus, $\{|\xi|^{b+a|\beta|}D_\xi^\beta[p_j(x+z, y, \xi)-p_j(x+z, 0, \xi)]\ \theta(2^{-j}\xi)\}$ is also a bounded subset of $S^0_{a,\delta}$, with bounds majorized by $(2^jt)^a$.

We have,

$$\begin{aligned}&\|(2^{ja}x)^\alpha[k_j(x+z, x-y)-k_j(x+z, x)]\|_2\\ &\leqslant C2^{ja|\alpha|}(2^jt)^a\sum_{\beta\leqslant\alpha}\||\xi|^{-a|\beta|-b}D^{\alpha-\beta}\psi(2^{-j}\xi)\|_2\\ &\leqslant C\cdot 2^{jan/2}(2^jt)^a\end{aligned}$$

and consequently,

$$\begin{aligned}&\int_{|x|\geqslant 2t^a}|k_j(x+z, x-y)-k_j(x+z, x)|\,dx\\ &\qquad\leqslant C2^{jan/2}(2^jt)^a\left[\int(1+2^{2aj}|x|^2)^{-N}\,dx\right]^{1/2}\leqslant C(2^jt)^a,\end{aligned}$$

$$\begin{aligned}&\int_{|x|\geqslant 2t^a}|K_m(x+z, x-y)-K_m(x+z, x)|\,dx\\ &\qquad\leqslant\sum_{j=0}^{m}\min((2^jt)^a, (2^jt)^{a(n/2-N)}).\end{aligned}$$

If $t\geqslant 1$, the right-hand side can be majorized by $\sum_{j=0}^\infty 2^{ja(n/2-N)}<\infty$. Suppose $t<1$, let j_0 such that $2^{j_0}t<1$, $2^{j_0+1}t\geqslant 1$.

We can estimate the right-hand side by

$$\sum_{j=0}^{j_0}2^{a(j-j_0)}+\sum_{j=j_0+1}^{\infty}(2^{j-j_0-1})^{a(n/2-N)}<\infty.$$

Since L^* is the operator defined by the amplitude $\overline{p(y, x, \xi)}$, the second part of (S_2') can be verified in the same way.

An integration by parts shows that $\mathscr{F}_\xi[p(x, y, \xi)](u)$ is a regular function for $|u| > 0$, so that we get the required representation of (Tf, g) when f, g have disjoint supports.

To verify (S_3), it suffices to recall that a pseudo-differential operator of order $-b$, maps L^2 into L_b^2, the Sobolev space of order b. Finally, if $p(x, y, \xi)$ is an amplitude supported on $\{|\xi| \leqslant 1\}$, L can be majorized by a convolution operator with kernel $C/(1+|x|)^m$, for an arbitrary m.

We would like to point out that most of the analysis above, was carried out supposing the amplitudes where C^∞ functions. Perhaps, more careful estimates would give the same results with less restrictive hypothesis of regularity.

The condition $L^*(1) = 0$ for a pseudo-differential operator L^* defined by a symbol $q(x, \xi)$, simply means that $L^*(1) = q(x, 0) = 0$. Thus, the condition can be stated explicitly. Note that L^* can be defined by a symbol whenever L is a properly supported operator, at least when $\delta < \rho$.

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